

# Permanence and extinction of regime-switching predator-prey models

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## Abstract

In this work we study the permanence and extinction of a regime-switching predator-prey model with Beddington-DeAngelis functional response. The switching process is used to describe the random changing of corresponding parameters such as birth and death rates of a species in different environments. Our criteria can justify whether a prey die out or not when it will die out in some environments and will not in others. Our criteria are rather sharp, and they cover the known on-off type results on permanence of predator-prey models without switching. Our method relies on the recent study of ergodicity of regime-switching diffusion processes.

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## 1 Introduction

In ecosystem, all species evolve and compete to seek resources to sustain their existence. Denote the two population sizes at time  $t$  by  $X_t$  and  $Y_t$ .  $X_t$  denotes the population size of the prey and  $Y_t$  stands for the population size of the predator. The Kolmogorov predator-prey model is a general deterministic model taking the form:

$$\begin{cases} \dot{X}_t = X_t f(X_t, Y_t), \\ \dot{Y}_t = Y_t g(X_t, Y_t). \end{cases}$$

For  $f(x, y) = b - py$  and  $g(x, y) = cx - d$ , one gets the well-known Lotka-Volterra model. These deterministic models have been extensively studied. We refer to the monograph [7] due to Hofbauer and Sigmund for related study on this deterministic model. Here  $f(X, Y)$  and  $g(X, Y)$  stand for the capita growth rate of each species, which is dependent on the population sizes of both species. Various functional response functions have been used considerably in modeling population dynamics such as Holling types, Hassel-Varley type, Leslie-Gower Holling II type, Beddington-DeAngelis type, etc. However, due to the continuous fluctuation in the environment, the birth rates, death rates, carrying capacities, competition coefficients and all other parameters involved in this model exhibit random fluctuation to a great extent. To describe this phenomenon, stochastic predator-prey models with different kinds of responses are proposed and there are many works to study these models (cf. [1], [9], [10] and references therein). For a predator-prey system with Beddington-DeAngelis functional response, in practice, we usually estimate the birth rate of the prey and death rate of the predator by their average values plus errors which follow normal distributions. As a result, one gets the following stochastic predator-prey model with Beddington-DeAngelis functional response:

$$\begin{cases} dX_t = X_t \left( a_1 - b_1 X_t - \frac{c_1 Y_t}{m_1 + m_2 X_t + m_3 Y_t} \right) dt + \alpha X_t dB_1(t), \\ dY_t = Y_t \left( -a_2 - b_2 Y_t + \frac{c_2 X_t}{m_1 + m_2 X_t + m_3 Y_t} \right) dt + \beta Y_t dB_2(t), \end{cases} \quad (1.1)$$

with  $X_0 = x_0 > 0$  and  $Y_0 = y_0 > 0$ . All the parameters in (1.1) are positive, and  $(B_1(t))$ ,  $(B_2(t))$  are Brownian motions on the line. For the model (1.1), Ji-Jiang [9] and Du et al. [6] studied the permanence and ergodicity, Liu-Wang [11] discussed stochastically asymptotic stability, and Li-Zhang [13] investigated non-persistence and strong/weak persistence.

From the viewpoint of biological modeling, variability of the environment may have important impact on the dynamics of the community. For instance, the distinctive seasonal change such as dry and rainy seasons are observed in monsoon forest, and it characterizes the vegetation there. Also, in boreal and arctic regions, seasonality exerts a strong influence on the dynamics of mammals. Moreover, the growth rates, the death rates and the carrying capacities often vary according to the changes in nutrition and food resources. All of these changes usually cannot be described by the traditional deterministic or stochastic predator-prey models. Therefore, it is natural to consider the predator-prey model in a random environment, which is formulated by an additional factor process. More precisely, consider the following regime-switching predator-prey model with Beddington-deAngelis functional response:

$$\begin{cases} dX_t = X_t \left( a_1(\Lambda_t) - b_1(\Lambda_t) X_t - \frac{c_1(\Lambda_t) Y_t}{m_1(\Lambda_t) + m_2(\Lambda_t) X_t + m_3(\Lambda_t) Y_t} \right) dt + \alpha(\Lambda_t) X_t dB_1(t) \\ dY_t = Y_t \left( -a_2(\Lambda_t) - b_2(\Lambda_t) Y_t + \frac{c_2(\Lambda_t) X_t}{m_1(\Lambda_t) + m_2(\Lambda_t) X_t + m_3(\Lambda_t) Y_t} \right) dt + \beta(\Lambda_t) Y_t dB_2(t) \end{cases} \quad (1.2)$$

with  $X_0 = x_0 > 0$  and  $Y_0 = y_0 > 0$ , where  $(B_1(t))$  and  $(B_2(t))$  are Brownian motions on the line, and  $(\Lambda_t)$  is a continuous time Markov chain with a finite state space  $\mathcal{S} = \{1, 2, \dots, N\}$ ,

$1 \leq N < \infty$ . Throughout this paper, the processes  $(B_1(t))$ ,  $(B_2(t))$  and  $(\Lambda_t)$  are defined on a complete probability space  $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, \mathbb{P})$ , and  $(\Lambda_t)$  is independent of  $(B_1(t))$  and  $(B_2(t))$ . The parameters  $a_k(\cdot)$ ,  $b_k(\cdot)$ ,  $c_k(\cdot)$  for  $k = 1, 2$ , and  $m_l(\cdot)$  for  $l = 1, 2, 3$ , are all positive functions on  $\mathcal{S}$ .

The dynamic system (1.2) is a regime-switching diffusion process, which has been widely applied in control problems, storage modeling, neutral activity, biology and mathematical finance. We refer the readers to [2, 4, 18, 19, 21, 22, 23] and the monographs [16, 25] for the study on recurrence, ergodicity, stability, numerical approximation of regime-switching diffusion processes with Markovian switching or state-dependent switching in a finite state space or an infinite state space. There is a vast literature on population dynamics with regime switching. For instance, Du-Du [5] described the omega-limit set of Kolmogorov systems of competitive type under the telegraph noise and investigated properties of stationary density; Zhu-Yin [27] examined certain long-run-average limits, and Zhu-Yin [28] investigated long-time behavior of sample paths for competitive Lotka-Volterra ecosystems. In the study of a population system, permanence and extinction are two important and interesting properties, respectively meaning that the population system will survive or die out in the future. Yuan et al. [26] discussed extinction for stochastic hybrid delay population dynamics with  $n$  interacting species, and Li et al. [14] and Liu-Wang [12] studied permanence and extinction for stochastic logistic populations with single species.

The regime-switching predator-prey model can describe a very important and interesting situation. We consider a simple example to introduce it. Let us consider the case  $\mathcal{S} = \{1, 2\}$ , where “1” denotes the rainy season and “2” denotes the dry season. It is rather possible to occur the situation that  $a_1(1) - \frac{1}{2}\alpha^2(1) > 0$  and  $a_1(2) - \frac{1}{2}\alpha^2(2) < 0$ . This means that the birth rate with perturbation of  $(X_t)$  in the rainy season makes sure that  $(X_t)$  will not die out, but that of  $(X_t)$  in the dry season makes  $(X_t)$  to die out. Then a natural question arises: will  $(X_t)$  in model (1.2) die out or not? This question is very interesting and represents an essential advantage of model (1.2) than model (1.1). However, the solution of this question is not easy, and so far there is few result of this type on the regime-switching predator-prey model or regime-switching population dynamics. On this topic, we refer the reader to [18] for the explicit examples to see the complexity of the regime-switching diffusion processes, and to [4, 19, 20, 23, 25] for some solutions of this type on ergodicity and stability of regime-switching diffusion processes. In this work, we shall provide a sharp criterion to justify whether  $(X_t)$  will die out or not (see Theorem 1.1 below).

Let  $(q_{ij})_{i,j \in \mathcal{S}}$  be the  $Q$ -matrix of the process  $(\Lambda_t)$  which means that

$$\mathbb{P}(\Lambda_{t+\delta} = l | \Lambda_t = k) = \begin{cases} q_{kl}\delta + o(\delta), & \text{if } k \neq l, \\ 1 + q_{kk}\delta + o(\delta), & \text{if } k = l, \end{cases} \quad (1.3)$$

for sufficiently small  $\delta > 0$ . Throughout this work, the matrix  $Q = (q_{ij})$  is assumed to be irreducible and conservative, i.e.  $q_{kk} = -q_k := -\sum_{j \neq k} q_{kj} < 0$ . As  $\mathcal{S}$  is a finite set and  $(q_{ij})$  is irreducible, the theory of Markov chains tells us that  $(\Lambda_t)$  is ergodic and there exists a unique stationary distribution  $(\mu_i)$  for it. To state our main result, we need to introduce two auxiliary processes. Let

$$d\varphi_t = \varphi_t(a_1(\Lambda_t) - b_1(\Lambda_t)\varphi_t)dt + \alpha(\Lambda_t)\varphi_t dB_1(t), \quad (1.4)$$

and

$$d\psi_t = \psi_t\left(-a_2(\Lambda_t) + \frac{c_2(\Lambda_t)}{m_2(\Lambda_t)} - b_2(\Lambda_t)\psi_t\right)dt + \beta(\Lambda_t)\psi_t dB_2(t) \quad (1.5)$$

with  $\varphi_0 = X_0 = x_0 > 0$  and  $\psi_0 = Y_0 = y_0 > 0$ , where  $(\Lambda_t)$ ,  $(B_1(t))$ ,  $(B_2(t))$  are defined as in (1.2). By comparison theorem of SDEs (cf. [8]),  $\varphi_t \geq X_t$  and  $\psi_t \geq Y_t$  a.s. for all  $t > 0$ . Our main result of this work is the following theorem.

**Theorem 1.1** *Let  $(X_t, Y_t, \Lambda_t)$  be defined by (1.2) and (1.3) and  $(\mu_i)_{i \in \mathcal{S}}$  be the stationary distribution of the process  $(\Lambda_t)$ .*

(i) *If  $\sum_{i \in \mathcal{S}} \mu_i(a_1(i) - \frac{1}{2}\alpha^2(i)) < 0$ , then  $\lim_{t \rightarrow \infty} X_t = 0$  a.s.,  $\lim_{t \rightarrow \infty} Y_t = 0$  a.s..*

(ii) *If  $\sum_{i \in \mathcal{S}} \mu_i(a_1(i) - \frac{1}{2}\alpha^2(i)) > 0$ , then  $(\varphi_t, \Lambda_t)$  is positive recurrent with stationary distribution  $\pi^\varphi$  on  $\mathbb{R}_+ \times \mathcal{S}$ . Assume further*

$$\lambda := -\sum_{i \in \mathcal{S}} \mu_i(a_2(i) + \frac{1}{2}\beta^2(i)) + \sum_{i \in \mathcal{S}} \int_{\mathbb{R}_+} \frac{c_2(i)x}{m_1(i) + m_2(i)x} \pi^\varphi(dx, i) < 0, \quad (1.6)$$

*then  $\lim_{t \rightarrow \infty} Y_t = 0$  a.s.,  $\limsup_{t \rightarrow \infty} X_t > 0$  a.s., and the distribution of  $(X_t, \Lambda_t)$  converges weakly to  $\pi^\varphi$ .*

(iii) *If  $\sum_{i \in \mathcal{S}} \mu_i(a_2(i) + \frac{1}{2}\beta^2(i) - \frac{c_2(i)}{m_2(i)}) < 0$ , then  $(\psi_t, \Lambda_t)$  is positive recurrent with stationary distribution  $\pi^\psi$ . Assume further that  $\sum_{i \in \mathcal{S}} \mu_i(a_1(i) - \frac{1}{2}\alpha^2(i)) > 0$  and*

$$\bar{\lambda} := \lambda + \sum_{i \in \mathcal{S}} \mu_i\left(a_2(i) + \frac{1}{2}\beta^2(i) - \frac{c_2(i)}{m_2(i)}\right) + \sum_{i \in \mathcal{S}} \int_{\mathbb{R}_+} b_2(i)y \pi^\psi(dy, i) > 0. \quad (1.7)$$

*Then  $\limsup_{t \rightarrow \infty} X_t > 0$  a.s.,  $\limsup_{t \rightarrow \infty} Y_t > 0$  a.s., and  $(X_t, Y_t, \Lambda_t)$  has a stationary distribution.*

This theorem will be proved in next section. As we mentioned, there are few results on permanence for the model (1.2) although there have been numerous works on extinction. Whereas, in Theorem 1.1, we provide a criteria which can justify whether a prey die out or not when it will die out in some environments and will not die out in other environments. We shall note that, when  $\mathcal{S}$  contains only one state, and hence there is no switching in (1.2) in this case, our result will coincide with the results in [6]. Actually, according to a similar calculation of [6, (2.3)],  $\sum_{i \in \mathcal{S}} \mu_i (a_2(i) + \frac{1}{2}\beta^2(i) - \frac{c_2(i)}{m_2(i)}) + \sum_{i \in \mathcal{S}} \int_{\mathbb{R}_+} b_2(i)y\pi^\psi(dy, i) = 0$  in the case  $N = 1$ , hence  $\bar{\lambda} = \lambda$  in this case. But when  $N > 1$ , we have no way to calculate the precise value of this term at present stage due to lack of explicit representation of the invariant measures of regime-switching diffusion processes. Moreover, we should point out that [13] investigated strong (weak) permanence *in the mean* for the model (1.1). Compared with [6], the main difficulty in present work is to determine the recurrent property of stochastic processes  $(\varphi_t, \Lambda_t)$  and  $(\psi_t, \Lambda_t)$ , which is overcome by using the recent results in [20] on justifying the recurrent properties of regime-switching diffusion processes. Besides, note that one more condition is needed in [6, Theorem 2.2]. Indeed, in the case  $\lambda > 0$ , another condition  $a_2 + \frac{1}{2}\beta^2 - \frac{c_2}{m_2} < 0$  should be added to guarantee the process  $(\psi_t)$  in [6] to be ergodic and hence their estimate (2.7) holds, i.e.  $\limsup_{t \rightarrow \infty} \frac{1}{t} \ln y(t) \leq 0$ . Without this condition, the estimate (2.7) in [6] is not correct.

The distributions  $\pi^\varphi$  and  $\pi^\psi$  are stationary distributions of one-dimensional regime-switching diffusion processes. In [21], for one-dimensional regime-switching diffusion process, an explicit representation of the stationary distribution is provided based on the hitting times of this process.

At last, according to the argument of our main result, the permanence or non-permanence of  $(X_t, Y_t)$  does not depend on the correlation of the Brownian motions  $(B_1(t))$  and  $(B_2(t))$  in the situation studied in this work. So in present work, we do not assume any condition on the dependence between  $(B_1(t))$  and  $(B_2(t))$ .

## 2 Proof of the main result

We first investigate the properties of the processes  $(\varphi_t, \Lambda_t)$  and  $(\psi_t, \Lambda_t)$  including the estimate of their moments and recurrent property. Set

$$\hat{a}_1 := \min_{i \in \mathcal{S}} a_1(i), \quad \check{a}_1 := \max_{i \in \mathcal{S}} a_1(i) \quad (2.1)$$

and similarly we can define  $\hat{a}_2, \check{a}_2, \hat{b}_k, \check{b}_k, \hat{c}_k, \check{c}_k$  for  $k = 1, 2$  and  $\hat{m}_l, \check{m}_l$  for  $l = 1, 2, 3$ . By the finiteness of  $\mathcal{S}$ , all the parameters appeared here remain to be positive.

**Lemma 2.1** For any  $p > 1$ ,

$$\begin{aligned}\mathbb{E}\varphi_t^p &\leq \left[ (\mathbb{E}\varphi_0^p)^{-\frac{1}{p}} \exp\left(-\left(\check{a}_1 + \frac{p-1}{2}\check{\alpha}^2\right)t\right) \right. \\ &\quad \left. + \frac{\hat{b}_1}{\check{a}_1 + \frac{p-1}{2}\check{\alpha}^2} \left(1 - \exp\left(-\left(\check{a}_1 + \frac{p-1}{2}\check{\alpha}^2\right)t\right)\right) \right]^{-p},\end{aligned}\tag{2.2}$$

and

$$\begin{aligned}\mathbb{E}\psi_t^p &\leq \left[ (\mathbb{E}\psi_0^p)^{-\frac{1}{p}} \exp\left(-\left(-\hat{a}_2 + \frac{\check{c}_2}{\hat{m}_2} + \frac{p-1}{2}\check{\beta}^2\right)t\right) \right. \\ &\quad \left. + \frac{\hat{b}_2}{-\hat{a}_2 + \frac{\check{c}_2}{\hat{m}_2} + \frac{p-1}{2}\check{\beta}^2} \left(1 - \exp\left(-\left(-\hat{a}_2 + \frac{\check{c}_2}{\hat{m}_2} + \frac{p-1}{2}\check{\beta}^2\right)t\right)\right) \right]^{-p}.\end{aligned}\tag{2.3}$$

Therefore, for any  $p > 1$ ,

$$\limsup_{t \rightarrow \infty} \mathbb{E}\varphi_t^p \leq \left( \frac{\hat{b}_1}{\check{a}_1 + \frac{p-1}{2}\check{\alpha}^2} \right)^{-p},\tag{2.4}$$

and for any  $p > 1$  such that  $-\hat{a}_2 + \frac{\check{c}_2}{\hat{m}_2} + \frac{p-1}{2}\check{\beta}^2 > 0$ , it holds

$$\limsup_{t \rightarrow \infty} \mathbb{E}\psi_t^p \leq \left( \frac{\hat{b}_2}{-\hat{a}_2 + \frac{\check{c}_2}{\hat{m}_2} + \frac{p-1}{2}\check{\beta}^2} \right)^{-p}.\tag{2.5}$$

**Proof.** We shall only prove the estimate for  $\varphi_t$  since the estimate for  $\psi_t$  can be done in the same way. By Itô's formula,

$$\begin{aligned}d\varphi_t^p &= p\left(a_1(\Lambda_t) + \frac{p-1}{2}\alpha^2(\Lambda_t)\right)\varphi_t^p dt - pb_1(\Lambda_t)\varphi_t^{p+1} dt + p\alpha(\Lambda_t)\varphi_t^p dB_1(t) \\ &\leq p\left(\check{a}_1 + \frac{p-1}{2}\check{\alpha}^2\right)\varphi_t^p dt - p\hat{b}_1\varphi_t^{p+1} dt + p\alpha(\Lambda_t)\varphi_t^p dB_1(t).\end{aligned}$$

Taking the expectation on both sides and utilizing Hölder's inequality yields that

$$\begin{aligned}\frac{d\mathbb{E}\varphi_t^p}{dt} &\leq p\left(\check{a}_1 + \frac{p-1}{2}\check{\alpha}^2\right)\mathbb{E}\varphi_t^p - p\hat{b}_1\mathbb{E}\varphi_t^{p+1} \\ &\leq p\left(\check{a}_1 + \frac{p-1}{2}\check{\alpha}^2\right)\mathbb{E}\varphi_t^p - p\hat{b}_1\left(\mathbb{E}\varphi_t^p\right)^{\frac{p+1}{p}}.\end{aligned}$$

By the comparison theorem of ordinary differential equation, we get (2.2), and then (2.4) follows immediately.  $\blacksquare$

By the Lemma 2.1, the processes  $(\varphi_t, \Lambda_t)$  and  $(\psi_t, \Lambda_t)$  are nonexplosive, and hence the processes  $(X_t, \Lambda_t)$  and  $(Y_t, \Lambda_t)$  are also nonexplosive by the comparison theorem.

**Lemma 2.2** *It holds that*

$$\mathbb{P}(X_t > 0, \forall t > 0) = 1 \quad \text{and} \quad \mathbb{P}(Y_t > 0, \forall t > 0) = 1. \quad (2.6)$$

**Proof.** We only consider the process  $(X_t)$  and the result for  $(Y_t)$  can be proved in the same way. Let

$$\tau_\Delta = \inf\{t > 0; X_t \leq \Delta\}, \quad \Delta > 0; \quad \sigma_K = \inf\{t > 0; X_t \geq K\}, \quad K \geq 1.$$

Let  $\tau_0 = \inf\{t > 0; X_t = 0\}$ , then  $\tau_\Delta \uparrow \tau_0$  as  $\Delta \downarrow 0$ . As  $(X_t, \Lambda_t)$  is nonexplosive,  $\lim_{K \rightarrow \infty} \sigma_K = \infty$  with probability 1. By Itô's formula, for  $p > 0$  such that  $-\hat{a}_1 + \frac{\check{c}_1}{\hat{m}_3} + \frac{p+1}{2}\check{\alpha}^2 > 0$ , we have

$$\begin{aligned} \mathbb{E}X_{t \wedge \sigma_K \wedge \tau_\Delta}^{-p} &= x_0^{-p} + p\mathbb{E} \int_0^{t \wedge \sigma_K \wedge \tau_\Delta} \left( -a_1(\Lambda_s) + \frac{p+1}{2}\alpha^2(\Lambda_s) \right. \\ &\quad \left. + \frac{c_1(\Lambda_s)Y_s}{m_1(\Lambda_s) + m_2(\Lambda_s)X_s + m_3(\Lambda_s)Y_s} + b_1(\Lambda_s)X_s \right) X_s^{-p} ds \\ &\leq x_0^{-p} + p\mathbb{E} \int_0^{t \wedge \sigma_K \wedge \tau_\Delta} \left( -\hat{a}_1 + \frac{\check{c}_1}{\hat{m}_3} + \frac{p+1}{2}\check{\alpha}^2 + \check{b}_1 K \right) X_s^{-p} ds. \end{aligned}$$

By Gronwall's inequality,

$$\mathbb{E}X_{t \wedge \sigma_K \wedge \tau_\Delta}^{-p} \leq x_0^{-p} \exp \left( p \left( -\hat{a}_1 + \frac{\check{c}_1}{\hat{m}_3} + \frac{p+1}{2}\check{\alpha}^2 + \check{b}_1 K \right) t \right). \quad (2.7)$$

If  $\mathbb{P}(\tau_0 < \infty) > 0$ , we can choose  $t, K$  large enough so that  $\mathbb{P}(\tau_0 < t \wedge \sigma_K) > 0$ . Then for any  $\Delta > 0$ ,

$$\begin{aligned} 0 &< \mathbb{P}(\tau_0 < t \wedge \sigma_K) \leq \mathbb{P}(\tau_\Delta < t \wedge \sigma_K) \\ &\leq \Delta^p \mathbb{E}[X_{t \wedge \sigma_K \wedge \tau_\Delta}^{-p} \mathbf{1}_{\tau_\Delta < t \wedge \sigma_K}] \leq \Delta^p \mathbb{E}[X_{t \wedge \sigma_K \wedge \tau_\Delta}^{-p}] \\ &\leq \Delta^p x_0^{-p} \exp \left( p \left( -\hat{a}_1 + \frac{\check{c}_1}{\hat{m}_3} + \frac{p+1}{2}\check{\alpha}^2 + \check{b}_1 K \right) t \right). \end{aligned}$$

Taking  $\Delta \downarrow 0$ , we get a contradiction. Hence  $\mathbb{P}(\tau_0 < \infty) = 0$  and we complete the proof.  $\blacksquare$

Next, we go to study the recurrent properties of the processes  $(\varphi_t, \Lambda_t)$  and  $(\psi_t, \Lambda_t)$ , which plays the fundamental role in the proof of our Theorem 1.1.

**Lemma 2.3 (Key lemma)** (i) *When  $\sum_{i \in \mathcal{S}} \mu_i(a_1(i) - \frac{1}{2}\alpha^2(i)) > 0$ , the process  $(\varphi_t, \Lambda_t)$  is positive recurrent and has a unique stationary distribution  $\pi^\varphi$ , which is a probability measure on  $(0, \infty) \times \mathcal{S}$ . When  $\sum_{i \in \mathcal{S}} \mu_i(a_1(i) - \frac{1}{2}\alpha^2(i)) < 0$ , the process  $(\varphi_t, \Lambda_t)$  is transient.*

(ii) *When  $\sum_{i \in \mathcal{S}} \mu_i(a_2(i) + \frac{1}{2}\beta^2(i) - \frac{c_2(i)}{m_2(i)}) > 0$ , the process  $(\psi_t, \Lambda_t)$  is positive recurrent and has a unique stationary distribution  $\pi^\psi$ , which is a probability measure on  $(0, \infty) \times \mathcal{S}$ . When  $\sum_{i \in \mathcal{S}} \mu_i(a_2(i) + \frac{1}{2}\beta^2(i) - \frac{c_2(i)}{m_2(i)}) < 0$ , the process  $(\psi_t, \Lambda_t)$  is transient.*

**Proof.** (1) We shall use the criterion established in [20] to prove this lemma. As the diffusion coefficient of  $\varphi_t$  is degenerate at the point  $x = 0$ , we use the transform  $Z_t = \ln \varphi_t$ . By Lemma 2.2, this transform makes sense for all  $t \geq 0$  a.s.. Then  $Z_t$  satisfies the following SDE:

$$dZ_t = (a_1(\Lambda_t) - \frac{1}{2}\alpha^2(\Lambda_t) - b_1(\Lambda_t)e^{Z_t})dt + \alpha(\Lambda_t)dB_1(t).$$

The recurrent property of  $(\varphi_t, \Lambda_t)$  is equivalent to that of  $(Z_t, \Lambda_t)$ . For each  $i \in \mathcal{S}$ , set  $L^{(i)} := (a_1(i) - \frac{1}{2}\alpha^2(i) - b_1(i)e^x)\frac{d}{dx} + \frac{1}{2}\alpha^2(i)\frac{d^2}{dx^2}$ . Then the operator  $\mathcal{A}$  defined by

$$\mathcal{A}f(x, i) = L^{(i)}f(\cdot, i)(x) + \sum_{j \neq i} q_{ij}(f(x, j) - f(x, i)), \quad f \in C^2(\mathbb{R} \times \mathcal{S})$$

is the infinitesimal generator of the process  $(Z_t, \Lambda_t)$ .

Set  $\beta_i = a_1(i) - \frac{1}{2}\alpha^2(i)$ ,  $i \in \mathcal{S}$ . We first prove the positive recurrence of  $(Z_t, \Lambda_t)$ . Take  $h(x) = x^2$  and  $g(x) = |x|$ . Then

$$L^{(i)}h(x) = 2(a_1(i) - \frac{1}{2}\alpha^2(i) - b_1(i)e^x + \frac{1}{2}\alpha^2(i)x^{-1})x, \quad \text{for } |x| > 0. \quad (2.8)$$

Note that  $\lim_{x \rightarrow +\infty} -b_1(i)e^x = -\infty$ ,  $\lim_{x \rightarrow -\infty} -b_1(i)e^x = 0$ . As  $\sum_{i \in \mathcal{S}} \mu_i \beta_i > 0$ , there exist  $\varepsilon > 0$  and  $r_0 > 0$  such that  $\sum_{i \in \mathcal{S}} \mu_i(\beta_i - \varepsilon) > 0$  and

$$2(a_1(i) - \frac{1}{2}\alpha^2(i) - b_1(i)e^x + \frac{1}{2}\alpha^2(i)x^{-1})x \leq -2(\beta_i - \varepsilon)g(x), \quad \text{for } x > r_0, i \in \mathcal{S}. \quad (2.9)$$

$$2(a_1(i) - \frac{1}{2}\alpha^2(i) - b_1(i)e^x + \frac{1}{2}\alpha^2(i)x^{-1})x \leq -2(\beta_i - \varepsilon)g(x), \quad \text{for } x < -r_0, i \in \mathcal{S}. \quad (2.10)$$

Since  $-2\sum_{i \in \mathcal{S}} \mu_i(\beta_i - \varepsilon) < 0$  and  $\lim_{|x| \rightarrow \infty} h(x) = \infty$ , By [20, Theorem 3.1], there exist a Lyapunov function  $V$  on  $\mathbb{R}_+ \times \mathcal{S}$  with  $V(x, i) \rightarrow \infty$  as  $|x| \rightarrow \infty$  and a constant  $r_0 > 0$  such that

$$\mathcal{A}V(x, i) < -1, \quad \forall |x| > r_0, i \in \mathcal{S}.$$

Consequently,  $(Z_t, \Lambda_t)$  and hence  $(\varphi_t, \Lambda_t)$  are positive recurrent. By [25, Theorem 4.3, pp.114],  $(Z_t, \Lambda_t)$  has a unique stationary distribution on  $\mathbb{R} \times \mathcal{S}$ . So the stationary distribution  $\pi^\varphi$  of  $(\varphi_t, \Lambda_t)$  is a probability measure on  $(0, \infty) \times \mathcal{S}$ .

Next, we prove the transience of  $(Z_t, \Lambda_t)$  under the condition that  $\sum_{i \in \mathcal{S}} \mu_i \beta_i < 0$ . We prove it directly due to the behavior of  $Z_t$  is quit different on  $(0, +\infty)$  and  $(-\infty, 0)$  caused by the term  $-b_1(i)e^x$ .

Let  $h(x) = \frac{1}{|x|}$  and  $g(x) = h'(x)$  for  $|x| > 0$ . Then

$$L^{(i)}h(x) = (a_1(i) - \frac{1}{2}\alpha^2(i) - b_1(i)e^x - \frac{\alpha^2(i)}{x})\frac{1}{x^2}, \quad \text{for } x < 0, i \in \mathcal{S}.$$



As

$$\lim_{x \rightarrow -\infty} a_1(i) - \frac{1}{2}\alpha^2(i) - b_1(i)e^x - \frac{\alpha^2(i)}{x} = \beta_i,$$

there exist  $\varepsilon > 0$ ,  $r_1 > 0$  so that

$$\sum_{i \in \mathcal{S}} \mu_i(\beta_i + \varepsilon) < 0, \quad \text{and } a_1(i) - \frac{1}{2}\alpha^2(i) - b_1(i)e^x - \frac{\alpha^2(i)}{x} < \beta_i + \varepsilon, \quad \forall x < -r_1, \quad i \in \mathcal{S}.$$

As  $\sum_{i \in \mathcal{S}} \mu_i(\beta_i + \varepsilon) < 0$ , the Fredholm alternative (cf. [17, pp.434]) yields that there exist a constant  $\kappa > 0$  and a vector  $\boldsymbol{\xi}$  such that

$$Q\boldsymbol{\xi}(i) = -\kappa - \beta_i - \varepsilon.$$

Set  $V(x, i) = h(x) + \xi_i g(x)$  for  $x < 0$ . We have

$$\begin{aligned} \mathcal{A}V(x, i) &= L^{(i)}h(x) + \xi_i L^{(i)}g(x) + Q\boldsymbol{\xi}(i)g(x) \\ &\leq \left( \beta_i + \varepsilon + \xi_i \frac{L^{(i)}g(x)}{g(x)} + Q\boldsymbol{\xi}(i) \right) g(x) \\ &= \left( -\kappa + \xi_i \frac{L^{(i)}g(x)}{g(x)} \right) g(x), \quad \text{for } x < -r_1, \quad i \in \mathcal{S}. \end{aligned}$$

It is easy to see that  $\lim_{x \rightarrow -\infty} \frac{L^{(i)}g(x)}{g(x)} = 0$ . Denote by  $\xi_{\min} = \min\{\xi_i; i \in \mathcal{S}\}$  and  $\xi_{\max} = \max\{\xi_i; i \in \mathcal{S}\}$ . Hence, there exists a constant  $r_2$  with  $r_2 > r_1 > 0$  such that  $h(x) + \xi_{\min}g(x)$  is an increasing function on  $(-\infty, -r_2]$  and

$$\mathcal{A}V(x, i) \leq 0, \quad \text{for } x \leq -r_2, \quad i \in \mathcal{S}.$$

Now, take  $Z_0 = z < -r_2 < 0$  such that

$$h(z) + \xi_{\max}g(z) < h(-r_2) + \xi_{\min}g(-r_2). \quad (2.11)$$

Set

$$\tau_{-K} = \inf\{t > 0; Z_t = -K\}, \quad \tau = \inf\{t > 0; Z_t = -r_2\}.$$

By Dynkin's formula, we have

$$\begin{aligned} \mathbb{E}[V(Z_{t \wedge \tau_{-K} \wedge \tau}, \Lambda_{t \wedge \tau_{-K} \wedge \tau})] &= V(z, i_0) + \mathbb{E} \int_0^{t \wedge \tau_{-K} \wedge \tau} \mathcal{A}V(Z_s, \Lambda_s) ds \\ &\leq V(z, i_0), \end{aligned}$$

where  $\Lambda_0 = i_0$ . Letting  $t \rightarrow \infty$ , we get

$$\mathbb{E}[V(-K, \Lambda_{\tau_{-K}}) \mathbf{1}_{\tau_{-K} \leq \tau}] + \mathbb{E}[V(-r_2, \Lambda_{\tau}) \mathbf{1}_{\tau < \tau_{-K}}] \leq V(z, i_0),$$

which yields further that

$$(h(-K) + \xi_{\min} g(-K))\mathbb{P}(\tau \geq \tau_{-K}) + (h(-r_2) + \xi_{\min} g(-r_2))\mathbb{P}(\tau < \tau_{-K}) \leq V(z, i_0),$$

$$\mathbb{P}(\tau \geq \tau_{-K}) \geq \frac{h(z) + \xi_{\max} g(z) - h(-r_2) - \xi_{\min} g(-r_2)}{h(-K) + \xi_{\min} g(-K) - h(-r_2) - \xi_{\min} g(-r_2)} > 0,$$

where in the last step we have used the increasing property of  $x \mapsto h(x) + \xi_{\min} g(x)$  on  $(-\infty, -r_2]$  and (2.11). Due to Lemma 2.1,  $\tau_{-K} \rightarrow \infty$  a.s. as  $K \rightarrow \infty$ . Consequently, passing to the limit as  $K \rightarrow \infty$ , the previous inequality yields

$$\mathbb{P}(\tau = \infty) > 0,$$

which implies that the process  $(Z_t, \Lambda_t)$  is transient, hence  $(\varphi_t, \Lambda_t)$  is transient too.

(2) The results of  $(\psi_t, \Lambda_t)$  can be proved by the method analogous to that used above for the process  $(\psi_t, \Lambda_t)$ , which is omitted.  $\blacksquare$

**Proposition 2.4 (Strong ergodicity theorem)** *Assume that  $(\varphi_t, \Lambda_t)$  and  $(\psi_t, \Lambda_t)$  are positive recurrent. Let  $f$  be a bounded measurable function on  $\mathbb{R} \times \mathcal{S}$ . Then almost surely*

$$\lim_{t \rightarrow \infty} \frac{1}{t} \int_0^t f(\varphi_s, \Lambda_s) ds = \sum_{i \in \mathcal{S}} \int_{\mathbb{R}_+} f(x, i) \pi^\varphi(dx, i), \quad (2.12)$$

and

$$\lim_{t \rightarrow \infty} \frac{1}{t} \int_0^t f(\psi_s, \Lambda_s) ds = \sum_{i \in \mathcal{S}} \int_{\mathbb{R}_+} f(y, i) \pi^\psi(dy, i). \quad (2.13)$$

Moreover,

$$\liminf_{t \rightarrow \infty} \frac{1}{t} \int_0^t b_1(\Lambda_s) \varphi_s ds \geq \sum_{i \in \mathcal{S}} \int_{\mathbb{R}_+} b_1(i) x \pi^\varphi(dx, i), \quad a.s., \quad (2.14)$$

and

$$\liminf_{t \rightarrow \infty} \frac{1}{t} \int_0^t b_2(\Lambda_s) \psi_s ds \geq \sum_{i \in \mathcal{S}} \int_{\mathbb{R}_+} b_2(i) y \pi^\psi(dy, i), \quad a.s.. \quad (2.15)$$

**Proof.** We shall only prove (2.12) and (2.14) for the process  $(\varphi_t, \Lambda_t)$  since (2.13) and (2.15) can be proved in the same way. The following idea goes back to that of [24, Theorem 3.16, pp.46]. For  $(x, i) \in (0, \infty) \times \mathcal{S}$ , let

$$\tau_{(x,i)} := \inf\{t > 0; (\varphi_t, \Lambda_t) = (x, i)\}.$$

For any  $(y, j) \in (0, \infty) \times \mathcal{S}$ ,  $(y, j) \neq (x, i)$ , as  $(\varphi_t, \Lambda_t)$  is positive recurrent, we have  $\mathbb{E}_{(x,i)} \tau_{(y,j)} < \infty$  and  $\mathbb{E}_{(y,j)} \tau_{(x,i)} < \infty$ . Denote by  $\zeta_1$  the time of first return to  $(x, i)$  after hitting  $(y, j)$ , and let

$\zeta_2 = \theta_{\zeta_1} \zeta_1$ , i.e.  $\zeta_2$  is the time interval between the first time  $(x, i)$  is hit after  $(y, j)$  has been visited and the second such hit. Denote by  $\zeta_n = \theta_{\sum_{k=1}^{n-1} \zeta_k} \zeta_1$ ,  $n \geq 3$ . By the time-homogeneous and strong Markovian properties of  $(\varphi_t, \Lambda_t)$ , we get that the variables  $(\zeta_k)$  are independent and identically distributed stopping times. Moreover,  $\mathbb{E}_{(x,i)} \zeta_1 = \mathbb{E}_{(x,i)} \tau_{(y,j)} + \mathbb{E}_{(y,j)} \tau_{(x,i)} < \infty$ .

Assume  $(\varphi_0, \Lambda_0) = (x, i)$  in the following. For a bounded measurable function  $f$  on  $(0, \infty) \times \mathcal{S}$ ,

$$\int_0^{\sum_{k=1}^n \zeta_k} f(\varphi_s, \Lambda_s) ds = \sum_{l=1}^n \int_{\sum_{k=1}^{l-1} \zeta_k}^{\sum_{k=1}^l \zeta_k} f(\varphi_s, \Lambda_s) ds.$$

Let

$$\eta_l := \int_{\sum_{k=1}^{l-1} \zeta_k}^{\sum_{k=1}^l \zeta_k} f(\varphi_s, \Lambda_s) ds.$$

Then  $(\eta_l)$  are mutually independent, and further

$$\mathbb{E}_{(x,i)} |\eta_l| \leq \|f\| \mathbb{E}_{(x,i)} \zeta_1 < \infty, \quad \text{where } \|f\| := \sup_{(z,k)} |f(z, k)|.$$

By virtue of the strong law of large numbers,

$$\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{l=1}^n \eta_l = \mathbb{E}_{(x,i)} \eta_1, \quad \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{l=1}^n \zeta_l = \mathbb{E}_{(x,i)} \zeta_1, \quad a.s.$$

Let  $\nu_t \in \mathbb{N}$  be such that  $\zeta_1 + \dots + \zeta_{\nu_t} \leq t < \zeta_1 + \dots + \zeta_{\nu_t+1}$ . Then, for bounded nonnegative measurable function  $f$ ,

$$\left( \sum_{l=1}^{\nu_t} \zeta_l \right)^{-1} \sum_{l=1}^{\nu_t+1} \eta_l \geq \frac{1}{t} \int_0^t f(\varphi_s, \Lambda_s) ds \geq \left( \sum_{l=1}^{\nu_t+1} \zeta_l \right)^{-1} \sum_{l=1}^{\nu_t} \eta_l.$$

It holds that  $\nu_t \rightarrow \infty$  as  $t \rightarrow \infty$  almost surely. Hence, the left-hand and right-hand sides of previous inequality both tend to  $(\mathbb{E}_{(x,i)} \zeta_1)^{-1} \mathbb{E}_{(x,i)} \eta_1$  almost surely. Since

$$\lim_{t \rightarrow \infty} \frac{1}{t} \int_0^t \mathbb{E}_{(x,i)} f(\varphi_s, \Lambda_s) ds = \sum_{k \in \mathcal{S}} \int_{\mathbb{R}_+} f(z, k) \pi^\varphi(dz, k),$$

it must hold that for any bounded nonnegative measurable function  $f$

$$\lim_{t \rightarrow \infty} \frac{1}{t} \int_0^t f(\varphi_s, \Lambda_s) ds = (\mathbb{E}_{(x,i)} \zeta_1)^{-1} \mathbb{E}_{(x,i)} \eta_1 = \sum_{k \in \mathcal{S}} \int_{\mathbb{R}_+} f(z, k) \pi^\varphi(dz, k) \quad a.s.. \quad (2.16)$$

By the linearity of  $f$  in (2.16), it is easy to show that (2.16) holds for all bounded measurable function  $f$ . So we have proved (2.12).

Applying (2.16) to function  $f(z, k) = b_1(k) \min\{z, M\}$  for positive constant  $M$ , we get

$$\lim_{t \rightarrow \infty} \frac{1}{t} \int_0^t b_1(\Lambda_s) \min\{\varphi_s, M\} ds = \sum_{k \in \mathcal{S}} \int_{\mathbb{R}_+} b_1(k) \min\{z, M\} \pi^\varphi(dz, k) \quad a.s..$$

Then

$$\begin{aligned} \liminf_{t \rightarrow \infty} \frac{1}{t} \int_0^t b_1(\Lambda_s) \varphi_s ds &\geq \lim_{t \rightarrow \infty} \frac{1}{t} \int_0^t b_1(\Lambda_s) \min\{\varphi_s, M\} ds \\ &= \sum_{k \in \mathcal{S}} \int_{\mathbb{R}_+} b_1(k) \min\{z, M\} \pi^\varphi(dz, k) \quad a.s.. \end{aligned}$$

Letting  $M$  tend to  $\infty$  in the previous inequality, we obtain (2.14). The proof is completed.  $\blacksquare$

After the preparation of above results on the auxiliary processes  $(\varphi_t)$  and  $(\psi_t)$ , we are ready to prove our main result. As the proof of Theorem 1.1 is a little long, we divide it into three propositions.

**Proposition 2.5** *If  $\sum_{i \in \mathcal{S}} \mu_i(a_1(i) - \frac{1}{2}\alpha^2(i)) < 0$ , then almost surely  $\lim_{t \rightarrow \infty} X_t = 0$  and  $\lim_{t \rightarrow \infty} Y_t = 0$ .*

**Proof.** By Itô's formula, we have

$$\begin{aligned} d \ln X_t &= \left( a_1(\Lambda_t) - \frac{1}{2} \alpha^2(\Lambda_t) - b_1(\Lambda_t) X_t - \frac{c_1(\Lambda_t) Y_t}{m_1(\Lambda_t) + m_2(\Lambda_t) X_t + m_3(\Lambda_t) Y_t} \right) dt + \alpha(\Lambda_t) dB_1(t) \\ &\leq \left( a_1(\Lambda_t) - \frac{1}{2} \alpha^2(\Lambda_t) \right) dt + \alpha(\Lambda_t) dB_1(t) =: d \ln \tilde{\varphi}_t. \end{aligned}$$

By comparison theorem of stochastic differential equation (cf. [8]), we have

$$\ln X_t \leq \ln \tilde{\varphi}_t \quad \text{for any } t > 0 \text{ a.s..}$$

Consequently,

$$\frac{\ln X_t}{t} \leq \frac{\ln \tilde{\varphi}_t}{t} = \frac{\ln x_0}{t} + \frac{1}{t} \int_0^t \left( a_1(\Lambda_s) - \frac{1}{2} \alpha^2(\Lambda_s) \right) ds + \frac{1}{t} \int_0^t \alpha(\Lambda_s) dB_1(s).$$

By the strong ergodicity theorem,

$$\lim_{t \rightarrow \infty} \left\{ \frac{1}{t} \int_0^t \left( a_1(\Lambda_s) - \frac{1}{2} \alpha^2(s) \right) ds + \frac{1}{t} \int_0^t \alpha(\Lambda_s) dB_1(s) \right\} = \sum_{i \in \mathcal{S}} \mu_i \left( a_1(i) - \frac{1}{2} \alpha^2(i) \right) \quad a.s..$$

Here we have used the fact

$$\lim_{t \rightarrow \infty} \frac{1}{t} \int_0^t \alpha(\Lambda_s) dB_1(s) = 0 \quad a.s.,$$

which is due to the boundedness of  $(\alpha(k))$  and the strong law of large numbers (cf. [15, Theorem 1.3.4]). Therefore, when  $\sum_{i \in \mathcal{S}} \mu_i(a_1(i) - \frac{1}{2}\alpha^2(i)) < 0$ , we have  $\limsup_{t \rightarrow \infty} \frac{\ln X_t}{t} < 0$  a.s., which further implies that  $\lim_{t \rightarrow \infty} X_t = 0$  a.s.

With  $\lim_{t \rightarrow \infty} X_t = 0$  a.s. in hand, we shall show that it must hold  $\lim_{t \rightarrow \infty} Y_t = 0$  a.s.. Namely, when the prey  $(X_t)$  dies out, the predator  $(Y_t)$  must die out too. By Itô's formula, it holds

$$\begin{aligned} \limsup_{t \rightarrow \infty} \frac{\ln Y_t}{t} &\leq \limsup_{t \rightarrow \infty} \left[ \frac{\ln y_0}{t} + \frac{1}{t} \int_0^t \left( -a_2(\Lambda_s) - \frac{\beta^2(\Lambda_s)}{2} - \frac{c_2(\Lambda_s)X_s}{m_1(\Lambda_s)} \right) ds + \frac{1}{t} \int_0^t \beta(\Lambda_s) dB_2(s) \right] \\ &= - \sum_{i \in \mathcal{S}} \mu_i(a_2(i) + \frac{1}{2}\beta^2(i)) < 0, \quad a.s.. \end{aligned}$$

which yields immediately that  $\lim_{t \rightarrow \infty} Y_t = 0$  a.s.. The proof is therefore completed.  $\blacksquare$

**Proposition 2.6** *If  $\sum_{i \in \mathcal{S}} \mu_i(a_1(i) - \frac{1}{2}\alpha^2(i)) > 0$ , and  $\lambda < 0$ , where  $\lambda$  is defined by (1.6), then almost surely  $\lim_{t \rightarrow \infty} Y_t = 0$ ,  $\limsup_{t \rightarrow \infty} X_t > 0$ , and the distribution of  $(X_t, \Lambda_t)$  converges weakly to  $\pi^\varphi$ .*

**Proof.** We first show that  $(Y_t)$  tends to be extinct. By Itô's formula, we have

$$d \ln Y_t = \left( -a_2(\Lambda_t) - \frac{1}{2}\beta^2(\Lambda_t) - b_2(\Lambda_t)Y_t + \frac{c_2(\Lambda_t)X_t}{m_1(\Lambda_t) + m_2(\Lambda_t)X_t + m_3(\Lambda_t)Y_t} \right) dt + \beta(\Lambda_t)dB_2(t).$$

Since  $X_t \leq \varphi_t$  a.s. for any  $t \geq 0$  and  $(\varphi_t, \Lambda_t)$  is positive recurrent thanks to Lemma 2.3, by the comparison theorem and Proposition 2.4, we obtain

$$\begin{aligned} \limsup_{t \rightarrow \infty} \frac{1}{t} \ln Y_t &\leq \limsup_{t \rightarrow \infty} \frac{1}{t} \int_0^t \left( -a_2(\Lambda_s) - \frac{1}{2}\beta^2(\Lambda_s) + \frac{c_2(\Lambda_s)\varphi_s}{m_1(\Lambda_s) + m_2(\Lambda_s)\varphi_s} \right) ds \\ &\quad + \limsup_{t \rightarrow \infty} \frac{1}{t} \int_0^t \beta(\Lambda_s) dB_2(s) \\ &= \lambda < 0, \quad a.s.. \end{aligned}$$

Therefore,  $\lim_{t \rightarrow \infty} Y_t = 0$  a.s..

Next, as  $\lim_{t \rightarrow \infty} Y_t = 0$  a.s., for any  $\varepsilon > 0$  there exist a measurable subset  $\Omega_\varepsilon \subset \Omega$  with  $\mathbb{P}(\Omega_\varepsilon) > 1 - \varepsilon$  and a positive constant  $t(\varepsilon)$  such that for any  $t > t(\varepsilon)$ ,  $\omega \in \Omega_\varepsilon$ ,

$$\begin{aligned} dX_t &\geq X_t \left( a_1(\Lambda_t) - b_1(\Lambda_t)X_t - \frac{c_1(\Lambda_t)\varepsilon}{m_1(\Lambda_t) + m_2(\Lambda_t)X_t + m_3(\Lambda_t)\varepsilon} \right) dt + \alpha(\Lambda_t)X_t dB_1(t) \\ dX_t &\leq X_t (a_1(\Lambda_t) - b_1(\Lambda_t)X_t) dt + \alpha(\Lambda_t)X_t dB_1(t), \end{aligned}$$

which yields that the distribution of  $(X_t, \Lambda_t)$  converges weakly to the stationary distribution  $\pi^\varphi$  of  $(\varphi_t, \Lambda_t)$ .

To complete the proof, we also need to show that  $\limsup_{t \rightarrow \infty} X_t > 0$  a.s.. Indeed, if  $\mathbb{P}(\lim_{t \rightarrow \infty} X_t = 0) > 0$ , there exists a measurable subset  $\Omega_0$  of  $\Omega$  such that  $\mathbb{P}(\Omega_0) > 0$  and  $\forall \omega \in \Omega_0, \lim_{t \rightarrow \infty} X_t(\omega) = 0$ . For any  $\delta > 0$ , as the distribution of  $(X_t, \Lambda_t)$  converges weakly to  $\pi^\varphi$ , we have

$$\pi^\varphi([0, \delta] \times \mathcal{S}) \geq \limsup_{t \rightarrow \infty} \mathbb{P}(X_t \in [0, \delta]) \geq \mathbb{P}(\Omega_0) > 0.$$

By the arbitrariness of  $\delta$ , we have  $\pi^\varphi(\{0\} \times \mathcal{S}) > \mathbb{P}(\Omega_0) > 0$  which contradicts with the fact that  $\pi^\varphi$  is a probability measure on  $(0, \infty) \times \mathcal{S}$  (see Lemma 2.3). We get the desired conclusion. ■

**Proposition 2.7** *If  $\sum_{i \in \mathcal{S}} \mu_i(a_1(i) - \frac{1}{2}\alpha^2(i)) > 0$ ,  $\sum_{i \in \mathcal{S}} \mu_i(a_2(i) + \frac{1}{2}\beta^2(i) - \frac{c_2(i)}{m_2(i)}) < 0$ , and  $\bar{\lambda} > 0$ , where  $\bar{\lambda}$  is defined by (1.7), then  $\limsup_{t \rightarrow \infty} X_t > 0$  a.s.,  $\limsup_{t \rightarrow \infty} Y_t > 0$  a.s. and  $(X_t, Y_t, \Lambda_t)$  has a stationary distribution.*

**Proof.** Since almost surely  $X_t \leq \varphi_t$  and  $Y_t \leq \psi_t$  for any  $t > 0$ , the strong ergodicity theorem yields that almost surely

$$\limsup_{t \rightarrow \infty} \frac{1}{t} \ln X_t \leq \limsup_{t \rightarrow \infty} \frac{1}{t} \ln \varphi_t \leq \sum_{i \in \mathcal{S}} \mu_i(a_1(i) - \frac{1}{2}\alpha^2(i)) - \sum_{i \in \mathcal{S}} \int_{\mathbb{R}_+} b_1(i)x \pi^\varphi(dx, i), \quad (2.17)$$

$$\begin{aligned} \limsup_{t \rightarrow \infty} \frac{1}{t} \ln Y_t &\leq \limsup_{t \rightarrow \infty} \frac{1}{t} \ln \psi_t \leq -\sum_{i \in \mathcal{S}} \mu_i(a_2(i) + \frac{1}{2}\beta^2(i) - \frac{c_2(i)}{m_2(i)}) \\ &\quad - \sum_{i \in \mathcal{S}} \int_{\mathbb{R}_+} b_2(i)y \pi^\psi(dy, i). \end{aligned} \quad (2.18)$$

Next, we apply the trick used in [6, Theorem 2.2] to estimate the lower bounds.

$$\begin{aligned} \frac{1}{t} \ln X_t &= \frac{1}{t} \ln x_0 + \frac{1}{t} \int_0^t (a_1(\Lambda_s) - \frac{1}{2}\alpha^2(\Lambda_s) - b_1(\Lambda_s)\varphi_s) ds \\ &\quad + \frac{1}{t} \int_0^t \left( b_1(\Lambda_s)(\varphi_s - X_s) - \frac{c_1(\Lambda_s)Y_s}{m_1(\Lambda_s) + m_2(\Lambda_s)X_s + m_3(\Lambda_s)Y_s} \right) ds + \frac{1}{t} \int_0^t \alpha(\Lambda_s) dB_1(s) \\ &\geq \frac{1}{t} \ln x_0 + \frac{1}{t} \int_0^t (a_1(\Lambda_s) - \frac{1}{2}\alpha^2(\Lambda_s) - b_1(\Lambda_s)\varphi_s) ds \\ &\quad + \frac{1}{t} \int_0^t \left( b_1(\Lambda_s)(\varphi_s - X_s) - \frac{c_1(\Lambda_s)Y_s}{m_1(\Lambda_s)} \right) ds + \frac{1}{t} \int_0^t \alpha(\Lambda_s) dB_1(s), \quad a.s.. \end{aligned}$$

Combining this with (2.17), we get

$$\liminf_{t \rightarrow \infty} \frac{1}{t} \int_0^t \left( b_1(\Lambda_s)(X_s - \varphi_s) + \frac{c_1(\Lambda_s)Y_s}{m_1(\Lambda_s)} \right) ds \geq 0, \quad a.s.,$$

and further

$$\liminf_{t \rightarrow \infty} \frac{1}{t} \int_0^t \left( \hat{b}_1(X_s - \varphi_s) + \frac{\check{c}_1}{\hat{m}_1} Y_s \right) ds \geq 0, \quad a.s.. \quad (2.19)$$

For the process  $(Y_t)$ , we have

$$\begin{aligned} \frac{1}{t} \ln Y_t &= \frac{1}{t} \ln y_0 - \frac{1}{t} \int_0^t b_2(\Lambda_s) Y_s ds - \frac{1}{t} \int_0^t \left( a_2(\Lambda_s) + \frac{1}{2} \beta^2(\Lambda_s) - \frac{c_2(\Lambda_s) \varphi_s}{m_1(\Lambda_s) + m_2(\Lambda_s) \varphi_s} \right) ds \\ &\quad - \frac{1}{t} \int_0^t \left( \frac{c_2(\Lambda_s) \varphi_s}{m_1(\Lambda_s) + m_2(\Lambda_s) \varphi_s} - \frac{c_2(\Lambda_s) X_s}{m_1(\Lambda_s) + m_2(\Lambda_s) X_s} \right) ds \\ &\quad - \frac{1}{t} \int_0^t \left( \frac{c_2(\Lambda_s) X_s}{m_1(\Lambda_s) + m_2(\Lambda_s) X_s} - \frac{c_2(\Lambda_s) X_s}{m_1(\Lambda_s) + m_2(\Lambda_s) X_s + m_3(\Lambda_s) Y_s} \right) ds \\ &\quad + \frac{1}{t} \int_0^t \beta(\Lambda_s) dB_2(s) \\ &\geq \frac{1}{t} \ln y_0 + \frac{1}{t} \int_0^t \left( -a_2(\Lambda_s) - \frac{1}{2} \beta^2(\Lambda_s) + \frac{c_2(\Lambda_s) \varphi_s}{m_1(\Lambda_s) + m_2(\Lambda_s) \varphi_s} \right) ds \\ &\quad - \frac{1}{t} \int_0^t \left( \frac{c_2(\Lambda_s)(\varphi_s - X_s)}{m_1(\Lambda_s) + m_2(\Lambda_s) \varphi_s} + \left( \frac{c_2(\Lambda_s) m_3(\Lambda_s)}{m_1(\Lambda_s) m_2(\Lambda_s)} + b_2(\Lambda_s) \right) Y_s \right) ds \\ &\quad + \frac{1}{t} \int_0^t \beta(\Lambda_s) dB_2(s), \quad a.s.. \end{aligned}$$

It follows that

$$\begin{aligned} \liminf_{t \rightarrow \infty} \frac{1}{t} \int_0^t &\left( \frac{c_2(\Lambda_s)(\varphi_s - X_s)}{m_1(\Lambda_s) + m_2(\Lambda_s) \varphi_s} + \left( \frac{c_2(\Lambda_s) m_3(\Lambda_s)}{m_1(\Lambda_s) m_2(\Lambda_s)} + b_2(\Lambda_s) \right) Y_s \right) ds \\ &\geq \limsup_{t \rightarrow \infty} \frac{1}{t} \int_0^t \left( -a_2(\Lambda_s) - \frac{1}{2} \beta^2(\Lambda_s) + \frac{c_2(\Lambda_s) \varphi_s}{m_1(\Lambda_s) + m_2(\Lambda_s) \varphi_s} \right) ds - \limsup_{t \rightarrow \infty} \frac{1}{t} \ln Y_t, \quad a.s.. \end{aligned} \quad (2.20)$$

Invoking (2.18) and by Proposition 2.4, Lemma 2.3, we get that

$$\begin{aligned} \liminf_{t \rightarrow \infty} \frac{1}{t} \int_0^t &\left( \frac{c_2(\Lambda_s)(\varphi_s - X_s)}{m_1(\Lambda_s) + m_2(\Lambda_s) \varphi_s} + \left( \frac{c_2(\Lambda_s) m_3(\Lambda_s)}{m_1(\Lambda_s) m_2(\Lambda_s)} + b_2(\Lambda_s) \right) Y_s \right) ds \\ &\geq \lambda + \sum_{i \in \mathcal{S}} \mu_i \left( a_2(i) + \frac{1}{2} \beta^2(i) - \frac{c_2(i)}{m_2(i)} \right) + \sum_{i \in \mathcal{S}} \int_{\mathbb{R}_+} b_2(i) y \pi^\psi(dy, i) = \bar{\lambda}, \quad a.s.. \end{aligned} \quad (2.21)$$

and further that

$$\liminf_{t \rightarrow \infty} \frac{1}{t} \int_0^t \left( \frac{\check{c}_2}{\hat{m}_1} (\varphi_s - X_s) + \left( \frac{\check{c}_2 \check{m}_3}{\hat{m}_1 \hat{m}_2} + \check{b}_2 \right) Y_s \right) ds \geq \bar{\lambda}, \quad a.s.. \quad (2.22)$$

Dividing both sides of (2.19) and (2.22) by  $\hat{b}_1$  and  $\check{c}_2/\hat{m}_1$  respectively, and adding them side by side, we obtain

$$\liminf_{t \rightarrow \infty} \frac{1}{t} \int_0^t Y_s ds \geq \frac{\hat{m}_1}{\check{c}_2} \left( \frac{\check{m}_3}{\hat{m}_2} + \frac{\check{b}_2 \hat{m}_1}{\check{c}_2} + \frac{\check{c}_1}{\hat{m}_1 \hat{b}_1} \right)^{-1} \bar{\lambda}, \quad a.s.. \quad (2.23)$$

Hence, if  $\bar{\lambda} > 0$ , then inequality (2.23) implies that it must hold  $\limsup_{t \rightarrow \infty} Y_t > 0$  a.s..

Note the following facts:

$$d \ln Y_t = \left( -a_2(\Lambda_t) - \frac{1}{2} \beta^2(\Lambda_t) + \frac{c_2(\Lambda_t) X_t}{m_1(\Lambda_t) + m_2(\Lambda_t) X_t + m_3(\Lambda_t) Y_t} \right) dt + \beta(\Lambda_t) dB_2(t),$$

and  $\lim_{t \rightarrow \infty} \frac{1}{t} \int_0^t \beta(\Lambda_s) dB_2(s) = 0$  a.s.. If  $\mathbb{P}(\{\omega : \lim_{t \rightarrow \infty} X_t(\omega) = 0\}) > 0$ , then it must hold

$$\mathbb{P} \left( \limsup_{t \rightarrow \infty} \frac{1}{t} \ln Y_t < 0 \right) > 0.$$

Hence, the fact  $\limsup_{t \rightarrow \infty} Y_t > 0$  a.s. implies that  $\limsup_{t \rightarrow \infty} X_t > 0$  a.s..

At last, we show the existence of invariant probability measure of the process  $(X_t, Y_t, \Lambda_t)$ . Note that almost surely  $X_t \leq \varphi_t$  and  $Y_t \leq \psi_t$  for any  $t > 0$ . By (2.4) and (2.5) of Lemma 2.1, we have that for  $p > 1$  satisfying  $-\hat{a}_2 + \frac{\check{c}_2}{\hat{m}_2} + \frac{p-1}{2} \check{\beta}^2 > 0$ , there exists a constant  $C > 0$  such that

$$\frac{1}{t} \int_0^t \mathbb{E}(X_s^p + Y_s^p + \Lambda_s^p) ds \leq C.$$

According to [3, Theorem 4.14], there exists a stationary distribution for  $(X_t, Y_t, \Lambda_t)$ . The proof is completed.  $\blacksquare$

### Proof of Theorem 1.1

The argument follows immediately from the arguments of Propositions 2.3, 2.5-2.7.  $\blacksquare$

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